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A GAME THEORETIC APPROACH - EVASION PROBLEM FOR A LINEAR SYSTEM WITH INTEGRAL CONSTRAINTS IMPOSED ON THE PLAYER CONTROL*

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A differential approach-evasion game with integral constraints imposed on the player controls is considered. A positional strategy is proposed, supplying a solution to the approach problem, and conditions are given under which the set of programmed absorption has the property of stability.

Differential games of this type were first studied in /1, 2/ where the pursuit-evasion problem was considered for single type systems, and auxiliary constructions were proposed for use as a basis in constructing a positional strategy in the form of the strategies of external aiming. Stable bridges of programmed absorption /3/ for linear, generally speaking, non-single type objects, were considered, and the construction of strategies extremal with respect to these bridges were demonstrated. A solution of the problem of approach in the class of positional procedures of control with a guide was proposed in /4/. It should be noted that the question of whether it is possible to construct resolving positional strategies in differential games with integral constraints, remains open.

The present paper is related to the investigations described in /1—8/.

1. Let a motion of a conflict-controlled system be described by the equation

$$\frac{d}{dt}x = A(t)x + B(t)u + C(t)v, \quad t \in [t_0, \vartheta], \quad x(t_0) = x_0 \tag{1.1}$$

Here x is an n-dimensional phase vector of the system, u and v are the controls of the first and second player respectively and represent the elements of the space R^p, and R^q, $A(t)$, $B(t)$, $C(t)$ are matrices of the corresponding dimensions depending continuously on t.

We assume that the player controls satisfy the following constraints

$$\int_{t_0}^{\vartheta} \| u[\tau] \|^2 d\tau \leqslant \mu_0^2, \quad \int_{t_0}^{\vartheta} \| v[\tau] \|^2 d\tau \leqslant \nu_0^2$$

and write

$$z_1[t] = \mu_0^2 - \int_{t_0}^{t} \| u[\tau] \|^2 d\tau, \quad z_2[t] = \nu_0^2 - \int_{t_0}^{t} \| v[\tau] \|^2 d\tau \tag{1.2}$$

We assume that every player knows the current position of the game $(n+3)$-dimensional vector $(t, z_1[t], z_2[t], x[t])$.

The problem facing the first player is that of choosing the control u which ensures that the point $z[\vartheta] = (z_1[\vartheta], z_2[\vartheta], x[\vartheta])$ arrives at the set $M^* = \{z\colon z = (z_1, z_2, x),\ z_1 \geqslant 0,\ z_2 \geqslant 0,\ x \in M\}$. Here M is a convex compactum belonging to R^n.

2. Let us list the necessary concepts and notation. We denote by U_{t_*, z_*}, V_{t_*, z_*}, $R_{(+)}^{n+2}$ the sets defined by the relations

$$R_{(+)}^{n+2} = \{z : z = (z_1, z_2, x), z_1 \geqslant 0, z_2 \geqslant 0, x \in R^n\}$$

$$U_{t_*, z_*} = \left\{u(\cdot) : u(\cdot) \in L_2[t_*, \vartheta], \int_{t_*}^{\vartheta} \| u(\tau) \|^2 d\tau \leqslant z_{1*}\right\}$$

$$V_{t_*, z_*} = \left\{v(\cdot) : v(\cdot) \in L_2[t_*, \vartheta], \int_{t_*}^{\vartheta} \| v(\tau) \|^2 d\tau \leqslant z_{2*}\right\}$$

Here $t_* \in [t_0, \vartheta]$, $z_* \in R_{(+)}^{n+2}$. We use the symbol

$$Z(t\colon t_*, z_*, v_*(\cdot)) \quad (t_* \in [t_0, \vartheta]$$
$$z_* \in R_{(+)}^{n+2}, \quad t \in [t_*, \vartheta,] \quad v_*(\cdot) \in V_{t_*, z_*})$$

*Prikl. Matem. Mekhan., 48, 4, 568-573, 1984 to denote the set of points

$$z = (z_1(t),\ z_2(t),\ x(t))$$

where

$$z_1(t) = z_{1*} - \int_{t_*}^{t} \| u(\tau) \|^2 d\tau, \quad z_2(t) = z_{2*} - \int_{t_*}^{t} \| v(\tau) \|^2 d\tau \tag{2.1}$$

$$x(t) = X[t, t_*] x_* + \int_{t_*}^{t} X[t, \tau] (B(\tau) u(\tau) + C(\tau) v(\tau)) d\tau$$

Here $X[t, \tau]$ is the fundamental matrix of system (1.1), $u(\tau)$ $(\tau \in [t_*, \vartheta])$ denote all possible functions of U_{t_*, z_*}.

Definition 2.1. We shall call the system of sets $\{W(t): W(t) \in R^{n+2}_{(+)}, t \in [t_0, \vartheta]\}$ a *u*-stable bridge if the following conditions hold:

1) $W(\vartheta) \in M^*$
2) the relation

$$Z(t^*:\ t_*, z_*, v_*(\cdot)) \cap W(t^*) \neq \varnothing.$$

holds for any instant of time t_* and t^* from $[t_0, \vartheta]$ $(t_* < t^*)$, any point $z_* \in W(t_*)$ and any function $v_*(\cdot) \in V_{t_*, z_*}$.

We will use the symbols T°, T_h $(h > 0)$ to denote the mappings of the set of all subsets of the space $R^{n+2}_{(+)}$ into itself, defined by the relations

$$T^\circ(G) = \bigcup_{\eta > 0} \overline{\mathrm{co}}\, G\eta, \quad T_h(G) = \bigcup_{z^* \in G} Z(z^*); \quad G \subseteq R^{n+2}_{(+)}$$

Here $\overline{\mathrm{co}}G_\eta$ is the closure of the convex shell /7/ of the set $G_\eta = \{z: z \in G,\ z_2 = \eta\}$, $Z(z^*) = \{z: z_1^* \leqslant z_1 \leqslant h,\ 0 \leqslant z_2 \leqslant z_2^*,\ x = x^*\}$. Let $\{W(t): t \in [t_0, \vartheta]\}$ be a *u*-stable bridge.

Assertion 2.1. The systems of sets

$$\{T^\circ(W(t)): t \in [t_0, \vartheta]\}, \quad \{T_h(W(t)): t \in [t_0, \vartheta]\} \quad (h > 0)$$

are *u*-stable bridges.

Below we shall assume that the folloiwng relation holds for the *u*-stable bridges in question:

$$T^\circ(W(t)) = W(t) \quad (t \in [t_0, \vartheta])$$

and, that we can find $h > \mu_0^2$ such that the relation

$$T_h(W(t)) = W(t) \quad (t \in [t_0, \vartheta])$$

holds. We also assume that the following assumptions hold.

Assumption A. The inequalities

$$\int_{t_*}^{t^*} \| l' H_i [t^*, \tau] \|^2 d\tau > 0, \quad i = 1, 2$$

where

$$H_1[t^*, \tau] = X[t^*, \tau] B(\tau)$$
$$H_2[t^*, \tau] = X[t^*, \tau] C(\tau)$$

hold for any unit vector l $(l \in R^n,\ \| l \| = 1)$ and any instant of time t_* and t^* from $[t_0, \vartheta]$ $(t_* < t^*)$.

Assumption B. A constant $\alpha > 0$ exists such, that for every instant of time τ from $[t_0, \vartheta]$ and any unit vector $n = (n_1, 0, n_{(3)})$ of the outer normal to the set $W_\eta(\tau)$ $(\eta \geqslant 0,\ W_\eta(\tau) \neq \varnothing)$, the inequality

$$n_1 \leqslant -\alpha$$

holds at the point z $(z \in \partial W(\tau),\ z_1 > 0)$. Here the symbol ∂W denotes the boundary of the set W.

Lemma 2.1. A constant $L > 0$ exists such, that for any instants of time t_* and t^* $(t_* < t^*)$ from $[t_0, \vartheta]$, any point z_* belonging to $W(t_*)$ and any function $v_*(\cdot) \in V_{t_*, z_*}$, a function $u(\cdot) \in U_{t_*, z_*}$ can be found such that

$$\int_{t_*}^{t^*} \| u(\tau) \|^2 d\tau \leqslant L(t^* - t_*)$$

and the inclusion

$$z(t^*;\ t_*, z_*;\ u(\cdot), v(\cdot)) \in W(t^*)$$

holds. Here $z(t^*:\ t_*, z_*:\ u(\cdot),\ v(\cdot)) = (z_1(t^*),\ z_2(t^*),\ x(t^*))$ is given by equations (2.1).

3. *Definition 3.1.* We define as the admissible positional strategy U of the first player, the mapping $U(t, z) = u_{t,z}(\cdot)$ of the positional space (t, z) $(t \in [t_0, \vartheta],\ z \in R_{(+)}^{n+2})$ onto the set $U_{t,z}$.

We denote by $\Gamma = \{\tau_i\colon t_0 = \tau_0 < \tau_1 < \ldots < \tau_s = \vartheta\}$ an arbitrary partition of the interval $[t_0, \vartheta]$.

Definition 3.2. We shall call the Euler polygonal line $y_\Gamma[t] = y_\Gamma[t\colon t_0, y_0, U, v[\cdot]]$ corresponding to the positional strategy U of the first player, and generated by the partition Γ and realization of the control $v[\cdot]$ of the second player $(v[\cdot] \in V_{t_0, y_0})$, the solution of the set of equations

$$\frac{d}{dt} z_{1\Gamma} = -\| U(\tau_i, y_i)\|^2, \quad \frac{d}{dt} z_{2\Gamma} = -\| v[t]\|^2$$
$$\frac{d}{dt} x_\Gamma = A(t) x_\Gamma + B(t) U(\tau_i, y_i) + C(t) v[t]$$
$$\tau_i \in \Gamma, \quad y_i = y_\Gamma[\tau_i] \quad (i = 0, 1, \ldots, s), \quad t \in [\tau_i, \tau_{i+1}]$$
$$(i = 0, 1, \ldots, s-1), \quad y_\Gamma(t_0) = y_0, \quad y = (z_1, z_2, x)$$

Definition 3.3. We shall call the function $x[t] = x[t\colon t_0, y_0, U]$ the motion of (1.1) corresponding to the admissible strategy of the first player and emerging from the initial point $y_0 = (\mu_0^2, \nu_0^2, x_0)$ at the initial instant t_0, provided that a sequence of Euler polygonal lines

$$y_{\Gamma(k)}[t] = y_{\Gamma(k)}[t : t_0 (\mu_0^2, \nu_0^2, x_k), U, v_k[\cdot]] \quad (k = 1, 2, \ldots)$$

can be found, satisfying the conditions

$$x_k \to x_0, \quad \max_{\tau_i, \tau_{i+1} \in \Gamma(k)} (\tau_{i+1} - \tau_i) \to 0,$$
$$\max_{t \in [t_0, \vartheta]} \| x_{\Gamma(k)}[t] - x[t]\| \to 0 \quad (k \to \infty)$$

The existence of the motions follows from Arzela's theorem /8/, since the set of functions $\{x_{\Gamma(k)}[\cdot]\}$ is uniformly equibounded and equicontinuous.

Let the position (t_*, z_*) satisfy the conditions

$$t_* \in [t_0, \vartheta], \quad z_* \in \{R_{(+)}^{n+2} \setminus W(t_*)\}, \quad W_{z_{2*}}(t_*) \neq \varnothing.$$
$$G \setminus F = \{z\colon z \in G, z \notin F\}$$

We consider the point z^* defined by the relation

$$\| z_* - z^* \| = \min_{z \in W_{z_{2*}}(t_*)} \| z_* - z\|$$

Since the set $W_{z_{2*}}(t_*)$ is convex, we have the relation $z_* - z^* = n^* \| z_* - z^* \|$; here $n^* = (n_1^*, 0, n_{(3)}^*)$ represents a suitable unit vector of the outer normal to the set $W_{z_{2*}}(t_*)$ at the point z^*.

Let us obtain the vector $u_*(t_*, z_*) = u_*$ from the relation

$$\min_{u \in R^p} n^{*\prime} \cdot f(t_*, z_*, u, v) = n^{*\prime} \cdot f(t_*, z_*, u_*, v) \qquad f(t, z, u, v) = \left\| \begin{matrix} -\|u\|^2 \\ -\|v\|^2 \\ A(t_*) x_* + B(t_*) u + C(t_*) v \end{matrix} \right\|$$

It can be shown that

$$u_x' = n^{*\prime} \cdot B(t_*)/n_1^*$$

By virtue of the assumption B a constant K $(K > 0)$, can be found such that the following inequality holds for any (t_*, z_*) $(t_* \in [t_0, \vartheta],\ z_* \in \{R_{(+)}^{n+2} \setminus W(t_*),\ z_1 > 0,\ W_{z_{2*}} = \varnothing\})$:

$$\| u_*(t_*, z_*)\| \leqslant K$$

Definition 3.4. We shall mean by the admissible extremal strategy $U^{(e)}$ of the first player, the following mapping $U^{(e)} = u_{t,z}(\cdot)$:

1) if $z \in W(t)$, then
$$u_{t,z}(\tau) = 0, \quad (\tau \in [t, \vartheta])$$

2) if $z \notin W(t)$, $W_{z_2}(t) \neq \varnothing$, then
$$u_{t,z}(\tau) = \begin{cases} u_*(t, z), & z_1 - \| u_*(t, z)\|^2 (\tau - t) \geqslant 0 \\ 0, & z_1 - \| u_*(t, z)\|^2 (\tau - t) < 0 \end{cases}$$
$$(\tau \in [t, \vartheta])$$

3) if $z \notin W(t)$, $W_{z_2}(t) = \varnothing$, then
$$u_{t,z}(\tau) = 0 \quad (\tau \in [t, \vartheta])$$

Theorem 3.1. If the initial state of the game is such that the point $y_0 = (\mu_0^2, \nu_0^2, x_0)$ lies within the set $W(t_0)$, the extremal strategy $U^{(e)}$ will bring any motion $x[t] = x[t\colon t_0, y_0, U^{(e)}]$

to the target set M at the instant ϑ.

The proof of the theorem is based on the estimation of the distance between the Euler polygonal line and u-stable bridge $\{W(t): t \in [t_0, \vartheta]\}$ in a Euclidean metric.

We will show that the Euler polygonal line generated by the partition Γ of the interval $[t_0, \vartheta]$ and the realization of the control $v[\cdot]$ of the second player $(v[\cdot] \in V_{t_0, (\mu_0^2, v_0^2, x)})$, satisfies one of the following inequalities:

$$\varepsilon^2 [x_\Gamma[\vartheta]] \leqslant \gamma$$
$$\varepsilon [x_\Gamma[\vartheta]] \leqslant \gamma^{1/2} \max_{t \in [t_0, \vartheta]} \| X[\vartheta, t] \| + \Big(\int_{t_0}^{\vartheta} \| H_1[\vartheta, \tau] \|^2 d\tau\Big)^{1/2} \{2K^2 \max_{\tau_i, \tau_{i+1} \in \Gamma} (\tau_{i+1} - \tau_i) + \gamma^{1/2}\}$$

where

$$\varepsilon [x_*] = \min_{m \in M} \| x_* - m \|$$
$$\gamma = [\| x_0 - x \|^2 + (\vartheta - t_0)\varphi] \exp [\beta(\vartheta - t_0)]$$
$$\beta = 2 \max_{t \in [t_0, \vartheta]} \| A(t) \|, \quad \varphi = \max_{\tau_i, \tau_{i+1} \in \Gamma} \varphi(\tau_{i+1} - \tau_i)$$

$\varphi(\delta)$ is a function continuous near 0, and $\lim \varphi(\delta) = 0$ $(\delta \to 0)$.

4. Let us introduce a new phase vector $x^{(1)}$ connected with the vector x by the relation

$$x^{(1)} = X[\vartheta, t]\, x \tag{4.1}$$

It can be shown that in the new variables the differential equation (1.1) will have the form

$$\frac{d}{dt} x^{(1)} = X[\vartheta, t] B(t) u + X[\vartheta, t] C(t) v \tag{4.2}$$
$$t \in [t_0, \vartheta], \quad x^{(1)}(t_0) = X[\vartheta, t_0]\, x_0$$

Since $X[\vartheta, \vartheta] = E$, it follows that the set M^* must be the target set for system (4.2). Relabelling the matrices $X[\vartheta, t] B(t), X[\vartheta, t] C(t)$, and the vector $x^{(1)}$ as $B(t), C(t)$ and x respectively, we obtain the initial system (1.1) where $A(t) = 0$ $(t \in [t_0, \vartheta])$.

We shall assume in subsequent discussions that the set M is defined by the relation

$$M = \{x: x \in R^n, \| x \| \leqslant d\} \quad (d \geqslant 0).$$

Let the following condition C hold: for any instants of time t_* and t^* $(t_0 \leqslant t_* < t^* \leqslant \vartheta)$, and any vector l $(l \in R^n, \| l \| = 1)$, the matrix functions $C(\cdot), B(\cdot)$ satisfy the inequality

$$\Big[\int_{t_*}^{\vartheta} \| l'C(\tau) \|^2 d\tau\Big]^{1/2} \int_{t^*}^{\vartheta} \| l'B(\tau) \|^2 d\tau \leqslant \Big[\int_{t^*}^{\vartheta} \| l'C(\tau) \|^2 d\tau\Big]^{1/2} \int_{t_*}^{\vartheta} \| l'B(\tau) \|^2 d\tau$$

Theorem 4.1. A system of sets $\{W(t: \vartheta, M^*): t \in [t_0, \vartheta]\}$ is the maximum u-stable bridge for the set M^*. Here

$$W(t: \vartheta, M^*) = \{z: z = (\mu^2, \nu^2, x), \quad \mu \geqslant 0, \quad \nu \geqslant 0, \quad x \in R^n_c$$
$$\max_{l \in R^n, \|l\| = 1} \Big[-\mu \Big(\int_t^{\vartheta} \| l'B(\tau) \|^2 d\tau\Big)^{1/2} + \nu \Big(\int_t^{\vartheta} \| l'C(\tau) \|^2 d\tau\Big)^{1/2} + l'x\Big] \leqslant d\Big\}$$

Its proof is based on the following lemma.

Lemma 4.1. Let $n = p = q = 1$ and let the function $C(\cdot), B(\cdot)$ satisfy at any instants of time t_* and t^* $(t_0 \leqslant t_* < t^* \leqslant \vartheta)$ the inequality

$$\Big[\int_{t_*}^{\vartheta} C^2(\tau)\, d\tau\Big]^{1/2} \int_{t^*}^{\vartheta} B^2(\tau)\, d\tau \geqslant \Big[\int_{t^*}^{\vartheta} C^2(\tau)\, d\tau\Big]^{1/2} \int_{t_*}^{\vartheta} B^2(\tau)\, d\tau$$

Then the system of sets $\{W(t: \vartheta, M^*): t \in [t_0, \vartheta]\}$ will be a maximum u-stable bridge for the set M^*.

The proof of the lemma is based on the proof of the relation

$$\partial W(t^*: \vartheta, M^*) \cap \{Z(t^*: t_*, z_*, v_*(\cdot)) \mid \operatorname{ri} Z(t^*: t_*, z_*, v_*(\cdot))\} \neq \varnothing$$

for any instants t_* and t^* $(t_0 \leqslant t_* < t^* \leqslant \vartheta)$, any point z_* $(z_* \in W(t_*: \vartheta, M^*))$ and an arbitrary function $v_*(\cdot)$ $(v_*(\cdot) \in V_{t_*, z_*})$.

Here $\operatorname{ri} Z$ denotes the relative interior /7/ of the set Z. The maximum property follows from the fact that $W(t: \vartheta, M^*)$ is the set of programmed absorption /3/ for M^* in the time interval $[t, \vartheta]$.

Condition C has the property of sufficiency, since if $B(t) = \text{const}$, $C(t) = \text{const}$, then we know that the system of sets $\{W(t: \vartheta, M^*): t \in [t_0, \vartheta]\}$ is a maximum u-stable bridge /3/ although condition C is clearly not satisfied.

Example. Let the dynamics of the controlled system be described by the equation

$$\dot{y}_1 = y_3, \quad \dot{y}_2 = y_4, \quad \dot{y}_3 = v_1, \quad \dot{y}_4 = v_2, \quad \dot{x}_1 = u_1, \quad \dot{x}_2 = u_2 \tag{4.3}$$

$$t \in [t_0, \vartheta], \quad y_i[t_0] = y_i^\circ, \quad x_j[t_0] = x_j^\circ \quad (i = 1, \ldots, 4;\ j = 1, 2)$$

Here $y = (y_1, y_2)$ are the coordinates of the material point $m^{(1)}$, (y_3, y_4) are the velocity components of this point, $v = (v_1, v_2)$ is the control applied to the inertial point $m^{(1)}$, $x = (x_1, x_2)$ are the coordinates of the inertialess point $m^{(2)}$ which is controlled by the choice of the velocity $u = (u_1, u_2)$. The player with control u must succeed in ensuring that the inequality

$$\| x[\vartheta] - y[\vartheta] \| \leqslant d \quad (d > 0)$$

holds at the instant ϑ whatever the method of control v. The following constraints are imposed on the realizations of player controls $u[\cdot], v[\cdot]$:

$$\int_{t_0}^{\vartheta} \| u[\tau] \|^2 d\tau \leqslant \mu_0^2, \quad \int_{t_0}^{\vartheta} \| v[\tau] \|^2 d\tau \leqslant \nu_0^2$$

Using the transformation (4.1) we change to the new variables

$$x_1^* = y_1 - x_1 + (\vartheta - t) y_3, \quad x_2 = y_2 - x_2 + (\vartheta - t) y_4$$

whose variations are described by the equations

$$\frac{d}{dt} x_1^* = (\vartheta - t) v_1 - u_1, \quad \frac{d}{dt} x_2^* = (\vartheta - t) v_2 - u_2 \tag{4.4}$$

$$x_1^*[t_0] = y_1^\circ - x_1^\circ + (\vartheta - t_0) y_3^\circ, \quad x_2^\circ[t_0] = y_2^\circ - x_2^\circ + (\vartheta - t_0) y_4^\circ$$

It is clear that the following relation holds:

$$\| x^*[\vartheta] \| = \| y[\vartheta] - x[\vartheta] \|, \quad x^* = (x_1^*, x_2^*).$$

It can be confirmed that system (4.4), equivalent to system (4.3), satisfies condition C.

In conclusion we note, that the system of sets $\{W(t\colon \vartheta, M^*)\colon t \in [t_0, \vartheta - \delta]\}$ $(0 < \delta < \vartheta - t_0)$ from Theorem 4.1 satisfies assumption B.

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